# Approximate Loop Transfer Recovery Method for Designing Fixed-Order Compensators

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**An approach is outlined for designing fixed-order dynamic compensators for multivariable time-invariant linear systems, based on minimizing a linear quadratic performance index. The formulation is done in an output feedback setting that exploits an observer canonical form to represent the compensator dynamics. The formulation also precludes the use of direct feedback of the plant output. The main contribution lies in defining a method for penalizing the states of the plant and of the compensator, and for choosing the distribution on initial conditions so that the loop transfer matrix approximates that of a full-state feedback design. When linear quadratic regulator theory is used to do the full-state feedback design, the approach can result in good gain and phase margin characteristics. Two examples are given to illustrate the effectiveness of the approach. The first treats the problem of pointing a flexible structure, and the second is a helicopter flight control problem using a tenth-order model for the fuselage and rotor dynamics. Both of the examples considered in this paper are for nonsquare plants.**

## Introduction

LINEAR quadratic regulator (LQR) synthesis methods have guaranteed stability margins. Unfortunately, this requires full-state feedback. It has been shown that the loop transfer properties of an LQR design for square, minimum phase plants can be recovered via an asymptotic design method.[1] This method relies on a cheap control formulation with a subset of the compensator dynamics becoming infinitely fast. It is often stated that the order of the compensator can later be reduced by discarding the fast modes; however, it is not clear how this can be accomplished without introducing direct feedthrough of the measured variables. It is generally good practice to avoid having direct feedthrough of sensor outputs to improve robustness and reduce the effect of sensor noise at high frequency. Aside from robustness issues, the order of the resulting compensator when designed for large-order systems may prove unwarranted.

Optimal output feedback design of fixed-order compensators,[2] introduced in the early 1970s, has received limited attention because of numerous difficulties associated with the design approach. Part of the difficulty lies in the fact that the compensator representation initially proposed was overparameterized; that is, the compensator formulation lacked a predefined structure, which invariably results in convergence problems when numerical optimization of the design is attempted. Several authors have adopted canonical structures that result in a minimal parameterization.[3,4] In this paper, the parameterization of Ref. 4 is used because it yields a convenient form when the problem is reformulated as a static gain optimal output feedback problem.

The other major objections to optimal output feedback design are that there are no guarantees on stability margins and few guidelines for penalizing plant states and compensator states to improve either performance or robustness. The major contribution in this paper is to present a formulation in which the objective of the fixed-order compensator design is to approximate the loop characteristics of a full-state design. Thus, much like the full-order compensator design case, a two-step design is implied—full-state feedback followed by approximate loop transfer recovery.

An outline of the paper is as follows. First, the problem formulation using the observer canonical structure of Ref. 4 is reviewed. Then, the procedure for penalizing the plant and compensator states and for selecting the initial distribution on the plant states to approximately recover the properties of a full-state design is presented. Finally, two design examples are presented. The first treats the problem of pointing a flexible structure, and the second is a helicopter flight control problem using a tenth-order model for the fuselage and rotor dynamics.

## Canonical Output Feedback Formulation

It has been shown[4] that for a multivariable system described by

$$\dot{x}_s = A_s x_s + B_s u, \qquad x_s \in R^n \tag{1}$$

$$y = C_s x_s + D_s u, \qquad y \in R^p \tag{2}$$

a fixed-order compensator without direct feedthrough of the output can be formulated in observer canonical form as

$$u = -H^0 z, \qquad u \in R^m \tag{3}$$

$$\dot{z} = P^0 z + u_c, \qquad z \in R^{n_c} \tag{4}$$

$$u_c = P_z u - N_y, \qquad u_c \in R^{n_c} \tag{5}$$

where

$$H^0 = \text{block diag } \{[0 \ldots 0\ 1]_{1 x \nu_i}\ i = 1, \ldots, m\} \tag{6}$$

$$P^0 = \text{block diag } [P_1^0, \ldots, P_m^0] \tag{7}$$

$$P_i^0 = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}_{\nu_i x \nu_i} \tag{8}$$

In Eqs. (3–5), $N$ and $P_z$ are free parameter matrices with dimensions $(n_c \times p)$ and $(n_c \times m)$, respectively. The dimensions of $H^0$ and $P^0$ are defined by the observability indices of the compensator, which are chosen to satisfy

$$\sum_{i=1}^{m} \nu_i = n_c, \qquad \nu_i \leq \nu_{i+1}$$

Received March 12, 1988; presented as Paper 88-4078 at the AIAA Guidance, Navigation, and Control Conference, Minneapolis, MN, Aug. 15–17, 1988; revision received Nov. 4, 1988. 

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The augmented system matrices

$$A=\begin{bmatrix} A_s & -B_sH^0 \\ 0 & P^0 \end{bmatrix} \qquad B=\begin{bmatrix} 0 \\ I_{nc} \end{bmatrix} \tag{9}$$

$$C=\begin{bmatrix} C_s & -DH^0 \\ 0 & H^0 \end{bmatrix} \qquad G=[N\,P_z] \tag{10}$$

define an optimal output feedback problem with the quadratic performance index

$$J=E_{x_0}\left\{\int_0^\infty [x^tQx+u_c^tRu_c]\,\mathrm{d}t\right\} \tag{11}$$

where the augmented state vector is

$$x^t=[x_s^t z^t] \tag{12}$$

The control $u_c$ in Eq. (4) is defined as

$$u_c=-GCx \tag{13}$$

and is used only in designing the compensator parameters, which are packed in the columns of $G$. The main advantage to this formulation lies in the fact that the problem has been converted to one of constant gain, limited state feedback, and the number of free parameters is the minimum needed to represent a strictly proper (but otherwise arbitrary) transfer function matrix. The necessary conditions for optimality[5,6] require the solution of the triple $\{G,K,L\}$ satisfying

$$A_c^tK+KA_c+Q+C^tG^tRGC=0 \tag{14}$$

$$A_cL+LA_c^t+X_0=0 \tag{15}$$

$$RGCLC^t-B^tKLC^t=0 \tag{16}$$

for a stable closed-loop system matrix

$$A_c=A-BGC \tag{17}$$

In Eq. (15),

$$X_0=E\{x_0x_0^T\} \tag{18}$$

is the variance matrix associated with the distribution assumed for the initial conditions. A block diagram illustrating the controller structure is given in Fig. 1.

One major disadvantage to an optimal output feedback formulation is that numerical methods must be used in solving Eqs. (14–16). A recent survey of computational methods for parametric LQ problems can be found in Ref. 7. Most of these methods generate a sequence of feedback gains ($G_k$) that, it is hoped, converge to a local minimizer of the cost function. The simplest case arises when all of the elements of $G$ are optimized; however, many of the methods can be extended to the case in which only some of the elements of $G$ are optimized, when an elimination of variables technique is used.[8] The most successful numerical methods are those that generate a sequence of monotonically decreasing cost function values. Associated with this problem is the choice of an initially stabilizing feedback gain ($G_0$), which is also addressed in Ref. 7. Because the solution is in general not unique, it is possible to converge to different locally minimizing solutions, depending on the choice of $G_0$. Thus, in using any particular method of solution, the designer should keep these issues in mind.

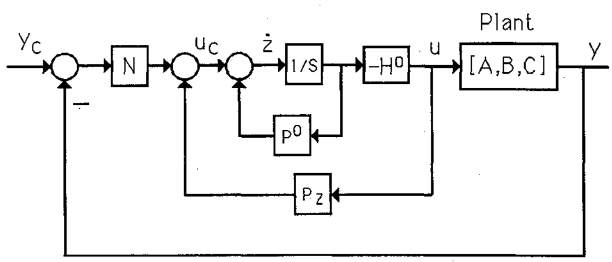



**Fig. 1 Controller structure in observer canonical form.**

For the examples presented in this paper, the solutions were obtained for a variety of starting values of $G_0$, and they were all found to converge to essentially the same solution. In general, our experience has been that rate of convergence and uniqueness of solution is strongly dependent on how the state and control weightings are chosen in Eq. (11). As a general observation, it was found that when the weightings were defined, as outlined in the next section, convergence was greatly improved over other (more arbitrary) methods of weight selection. The convergent algorithm given in Ref. 9 was used to generate the numerical results.

A second possible disadvantage lies in the fact that one does not know how to specify the order of the compensator at the outset. In addition, the observability indices in ($\nu_i$) influence the structure of the compensator. In general, we have found very little sensitivity to the choice of indices. Our general approach has been to start with the lowest possible order compensator ($n_c=m$), which requires $\nu_i=1$, and to observe the effect that increasing $n_c$ has on the state penalty portion of the index of performance. In the next section, we show that this represents a measure of the accuracy to which we are able to recover the loop characteristics of a full-state feedback design.

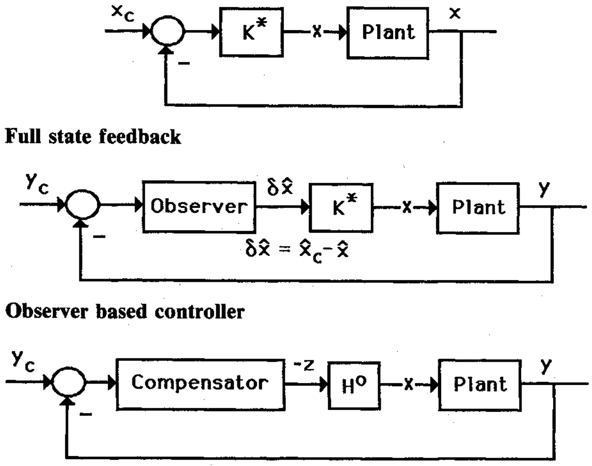
One advantage to the canonical form presented here is that $G$ is completely free; that is, all of the elements of $G$ are optimized, and an elimination of variables is not required. A second advantage is that all of the elements of $H^0$ are specified by the order of the compensator and the choice of observability indices. This is essential to the loop transfer recovery procedure to be presented in the next section.

## Loop Recovery Formulation

Full-state feedback design is often used as a first step in designing an output feedback controller for multivariable systems. A variety of methods exist, such as LQR theory, pole placement, eigenvalue/eigenvector assignment, model following control, and decoupling control design. The most popular method is LQR design. It is well known that this approach also yields guaranteed gain and phase margins when measured at the plant input.[10]

The full-state and output feedback control structures are illustrated in Fig. 2. The objective in observer-based controller design is to estimate the plant states and to use the estimated states in place of the actual states. This results in a higher order system where closed-loop eigenvalues and eigenvectors of the full-state design are preserved, and the compensator merely adds its own dynamics to the response. When the compensator is designed based on loop transfer recovery, it is also possible to recover the gain- and phase-margin properties of the full-state design. This amounts to suitably choosing the weighting matrices in a dual LQR formulation for the observer design. Both full-state and observer design problems are decoupled.

In fixed-order compensator design, the notion of state estimation is not present. However, it should be recognized that, as long as the loop transfer properties of a full-state design can be recovered to a sufficient degree of accuracy, the closed loop eigenvalues should contain a set of eigenvalues and eigenvectors that approximate those of the full-state design. More




a) **Full state feedback**

b) **Observer based controller**

c) **Fixed order controller**

**Fig. 2 Comparison of controller structures.**

important, the multivariable gain and phase margin properties should also be approximated. With this in mind, consider the loops for the full-state design and fixed-order controller design broken at the points marked $x$ in Fig. 2. The loop transfer properties of both loops are nearly equal when the time responses of the signals at the return of the loop are approximately equal for a set of identically chosen input signals, with zero initial conditions on all states in the two feedback systems.

The return signal in the case of full-state design is

$$u^* = -K^* x_s \tag{19}$$

Referring to Eq. (3), the return signal in the case of fixed-order compensator design is $-H^0 z$. Thus, the objective in designing the compensator should be to minimize

$$y_1 = K^* x_s - H^0 z \tag{20}$$

for a suitably chosen input and for zero initial conditions. Here we select the input waveforms as impulses, with magnitudes uniformly distributed on the unit sphere. This naturally leads to selecting the following index of performance:

$$J = E_{x_0}\left\{\int_0^\infty [y_1^t y_1 + \rho u_c^t u_c]\, \mathrm{d}t\right\} \tag{21}$$

Substituting for $y_1$ from Eq. (20) and rewriting Eq. (21) in the form of Eq. (11) lead to the following expressions for the weighting matrices:

$$Q = \begin{bmatrix} K^{*t}K^* & -K^{*t}H^0 \\ -H^{0t}K^* & H^{0t}H^0 \end{bmatrix} \qquad R = \rho I_m \tag{22}$$

For zero initial conditions, the effect of the impulses at the system input is to create an initial condition whose variance matrix is given by

$$X_0 = \begin{bmatrix} B_s B_s^t & 0 \\ 0 & 0 \end{bmatrix} \tag{23}$$

This is used in necessary condition (15) for the distribution on initial conditions. Note that, unlike the design of a full-order observer, the design of a fixed-order controller depends on the gain matrix from the full-state design step. Moreover, this gain matrix is not implemented as a part of the final controller (see Fig. 2). From Eqs. (22) and (23), it is apparent that the proposed loop recovery procedure can be viewed as a rationale for properly selecting the plant and compensator state weightings and the initial condition distribution matrix.

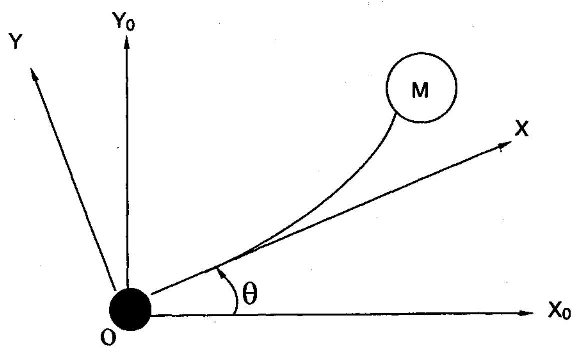



**Fig. 3 Geometry of the flexible arm model.**

In the preceding formulation, there is no requirement for a square plant. Also, the usual requirement that the plant be minimum phase is not explicit. However, it should be noted that this is not a true asymptotic recovery method in the sense that perfect recovery will take place in the limit as $\rho$ approaches zero. At best, it should be viewed as an approximation to the case of a full order observer, where $n_c$ will play a limiting factor. The same can be said for the case of full-order observer design, followed by observer order reduction. These two design steps are combined here in a single design procedure while maintaining a strictly proper compensator transfer function. The ability for loop recovery will also be limited if the plant has right-hand plane zeros, as will be illustrated in the second of the two examples that follow.

## Numerical Results

We present in this section two examples to illustrate the effectiveness of the design procedure. The first example treats a structural vibration model that arises in considering the rigid body and the first flexible mode of a lightweight flexible arm moving along a predefined path.[11] The second example is a helicopter flight control problem using a tenth-order model for the fuselage and rotor dynamics.[12]

### Example 1: Lightweight Flexible Arm

The dynamic model of the flexible arm prototype in the Flexible Automation Laboratory at Georgia Institute of Technology is used in this example. Figure 3 illustrates the geometry from which the model was derived. The arm moves on the horizontal plane and is stiff with respect to torsional effects. This flexible beam can also be seen as the last member of an open kinematic structure whose previous links are rigid. The derivation of the dynamic model for the flexible beam of Fig. 3 directly follows from Ref. 11. For the example considered here, the model is reduced to include the rigid body and the first flexible mode through residualization of the second flexible mode with a frequency equal to 87.5 rad/s. This results in a direct feedthrough of the input to the output. However, the form of the model is representative of the general problem of controller design for rapid pointing of a flexible structure. The control is motor torque in foot pounds.

The rigid body and the first flexible mode dynamics and outputs are defined by the following system matrices:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 11.92 & 0 & 0 \\ 0 & -192.9 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0.996 \\ -2.151 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.28\times 10^{-2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ -1.048\times 10^{-6} \\ 0 \end{bmatrix} \tag{24}$$

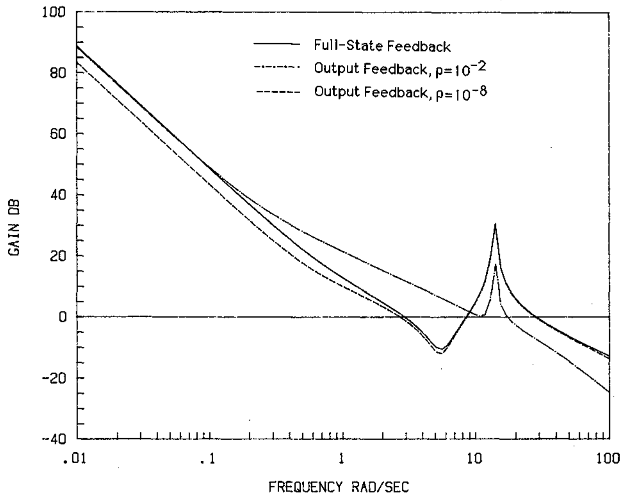



**Fig. 4 Loop gain for full-state and output feedback.**

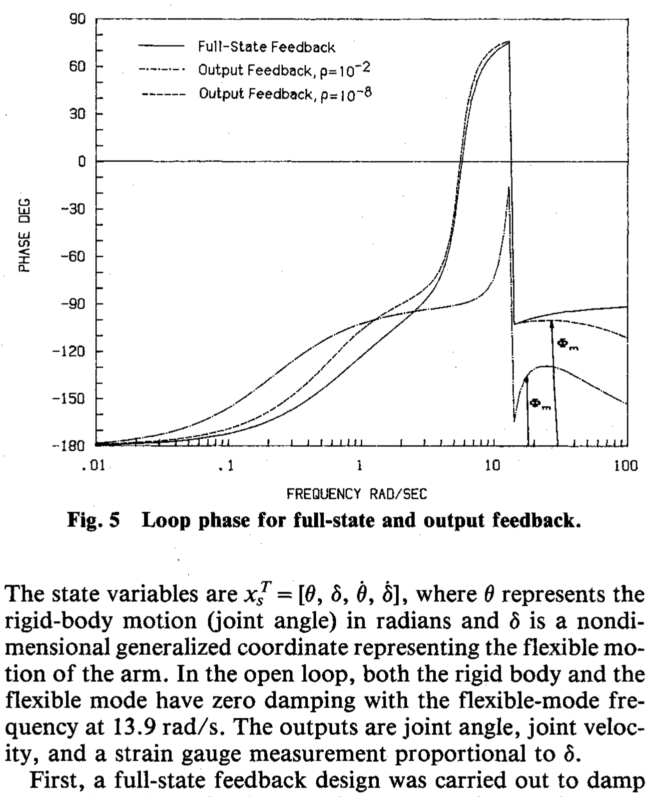



**Fig. 5 Loop phase for full-state and output feedback.**

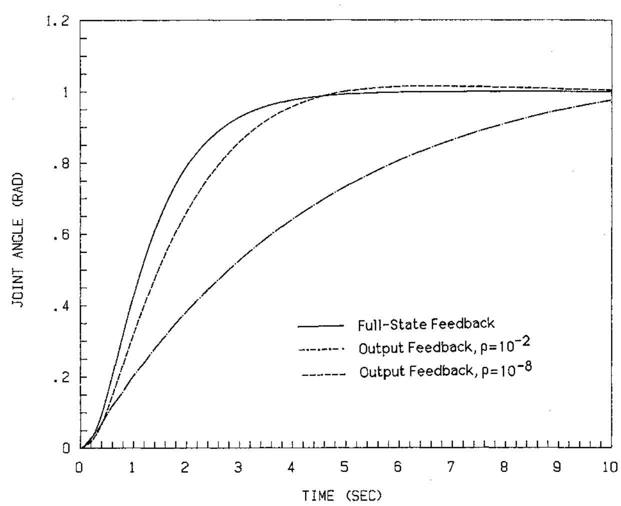



**Fig. 6 Joint angle time history for a step joint angle command input.**

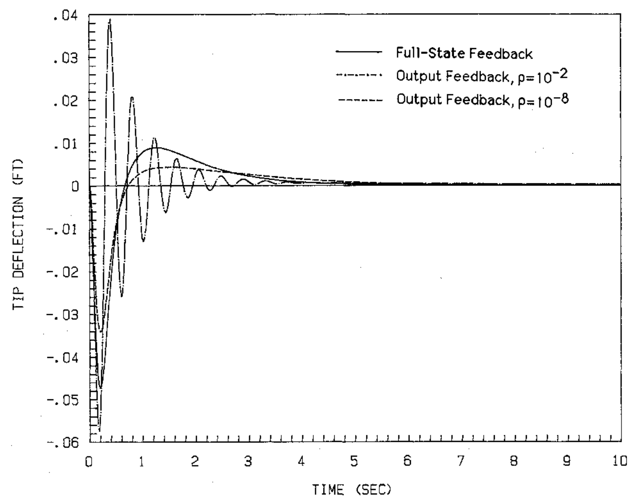



**Fig. 7 Tip deflection time history for a step joint angle command input.**

The state variables are $x_s^T = [\theta, \delta, \dot{\theta}, \dot{\delta}]$, where $\theta$ represents the rigid-body motion (joint angle) in radians and $\delta$ is a nondimensional generalized coordinate representing the flexible motion of the arm. In the open loop, both the rigid body and the flexible mode have zero damping with the flexible-mode frequency at 13.9 rad/s. The outputs are joint angle, joint velocity, and a strain gauge measurement proportional to $\delta$.

First, a full-state feedback design was carried out to damp the modes. The weighting matrices selected for the full-state feedback design are

$$Q = \text{diag}\ [10,\ 10,\ 10,\ 80], \qquad R = 1.0 \tag{25}$$

The full-state feedback design results in a damping ratio of 0.7 in the flexible mode and overdamped rigid-body modes with the following eigenvalues:

| | |
|---|---|
| Rigid body mode: | $-1.089,\ -2.618$ |
| Flexible mode: | $-9.438\ \pm 9.770j$ |

The gain matrix $K^*$ is

$$K^* = [3.162 \quad -28.507 \quad 4.434 \quad -8.446] \tag{26}$$

Next, an output feedback design with second-order compensation was carried out using the procedure described in the previous section. The weighting matrices $Q$ and $R$ in Eq. (22) were formulated using the full-state feedback gain matrix of Eq. (26). The value of $\rho$ in Eq. (22) was set to $1.0 \times 10^{-8}$. The output feedback design results in a damping ratio of roughly 0.75 in the flexible mode, with the gain matrix $G$ being

$$G = \left[\begin{array}{ccc|c} -4.27\times 10^2 & -1.15\times 10^8 & -7.11\times 10^2 & 2.52\times 10^2 \\ -5.13\times 10^2 & 2.41\times 10^6 & -5.32\times 10^3 & 2.35.\times 10^2 \end{array}\right] \tag{27}$$

The closed-loop eigenvalues with this gain matrix are

| | |
|---|---|
| Rigid-body mode: | $-0.583\ \pm 0.335j$ |
| Flexible mode: | $-9.947\ \pm 8.706j$ |
| Compensator: | $-4.119,\ -212.0$ |

A second design was done for $\rho = 0.01$. In this case, the damping of the flexible mode was only 0.1.

Figures 4 and 5 illustrate the recovery of the full-state design as $\rho$ is decreased. In addition to providing increased damping of the flexible mode, the $\rho = 10^{-8}$ design provides an additional 35 deg of phase margin at the high-frequency end. The command tracking performance of the full-state feedback controller is compared with the output feedback controller in Figs. 6 and 7. In Fig. 6, the joint angle time history for a step joint angle command input is shown for the full-state feedback design, as well as for the output feedback design. It is clear from this figure that, in spite of the fact that loop recovery was designed with the loop broken at the plant input, the output feedback controller is able to achieve output tracking performance close to the full-state feedback case. The resulting flexible tip deflection is presented in Fig. 7 for the same step joint angle command input, where the improved damping for $\rho = 10^{-8}$ is evident.

**Example 2: Helicopter Flight Control Problem**

A tenth-order model for the Sikorsky S-61 helicopter in hover flight condition is taken from Ref. 12, where a tight attitude control system is designed using full-state feedback. The model consists of fuselage longitudinal velocity, lateral velocity, pitch attitude, roll attitude, pitch rate, roll rate and two rigid degrees of freedom for the rotor, with body vertical and yawing motions decoupled from the rest of the dynamics. The quadratic performance index used was of the form

$$J=\frac{1}{2}\int_0^\infty [A_\theta(\theta_F^2+\phi_F^2)+A_u(u^2+v^2)+\theta_c^2+\theta_s^2]\ \mathrm{d}t \tag{28}$$

where $\theta_F$ and $\phi_F$ are body pitch and roll attitudes in radians, $u$ and $v$ are body longitudinal and lateral velocities in feet per second, and $\theta_c$ and $\theta_s$ are lateral and longitudinal cyclic inputs in radians. $A_\theta$ and $A_u$ are the weightings on the body attitudes and body velocities, respectively. To obtain a tight attitude command loop, the weightings on body roll and pitch attitudes were varied from $A_\theta = 1$–$100$. In Ref. 12, it was found that, in the absence of rotor state feedback, the rotor-mode response would become unstable as the control loop was tightened, and it was concluded that tight autopilot design should include rotor state feedback to insure stability of rotor response. In Ref. 12, this was accomplished through a full-order observer design and the resulting system was shown to be stable even when the attitude loop was tightened.

In this paper, we demonstrate the effectiveness of the new design procedure using body roll attitude, pitch attitude, roll rate, and pitch rates as outputs with a fourth-order compensator for the $A_\theta = 10$ case. The observability indices of the compensator are $\nu_1 = \nu_2 = 2$. The open-loop eigenvalues of the helicopter model are:

Mode 1: $-14.1 \pm 38.2j$ rotor nutation
Mode 2: $-13.2 \pm 5.2j$ rotor precision

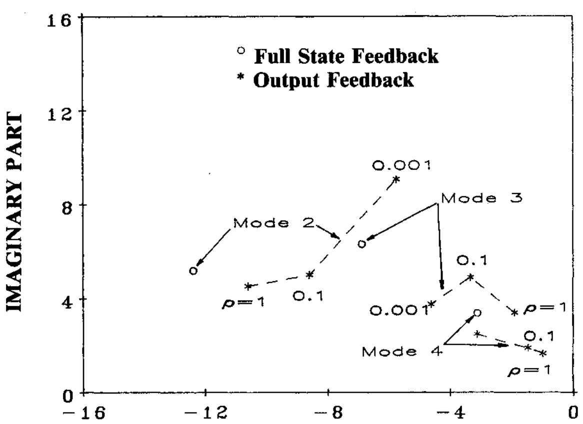



**Fig. 8 Comparison of closed loop modes.**

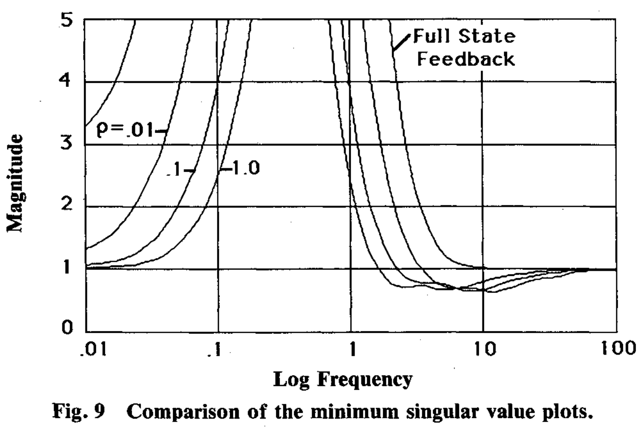



**Fig. 9 Comparison of the minimum singular value plots.**

Mode 3: $-1.20 \pm 0.21j$pitch/roll mode
Mode 4: $-0.11 \pm 0.36j$pitch/roll/longitudinal velocity mode
Mode 5: $-0.04 \pm 0.50j$pitch/roll/lateral velocity mode

First, a full-state feedback design was carried out using the weighting matrices in Ref. 12, for the $A_\theta = 10$ case. Using the gain values from the full-state feedback design, the weighting matrices $Q$ and $R$ in Eq. (22) were formulated for the output feedback design with compensator. The eigenvalues of the body pitch/roll/longitudinal velocity mode, the pitch/roll mode, and the lower-frequency rotor precision mode are shown in Fig. 8. The value of $\rho$ in Eq. (22) is varied from 1.0 to 0.01. Also shown are the full-state feedback design results for these modes. The rotor nutation mode (not shown) remains well damped and nearly identical to the open loop value for both designs. Note that mode 2 is departing from the full-state design value as $\rho$ is decreased. In this case, the plant is nonminimum phase. This can be seen by examining the transfer function from the input to the attitude outputs, which yields a pair of zeros at $(9.01 \pm i58.2)$. Hence, full recovery is not possible.

Figure 9 illustrates the degree of recovery that does take place, by comparing the minimum singular value plots of the return difference matrix for the loop broken at the input to the plant. Note that the minimum singular value in the vicinity of the mode 2 frequency does not increase above 0.65. However, the design procedure does attempt to approximate the performance and robustness of a full-state feedback design.

## Conclusions

A formulation has been presented for efficiently designing fixed-order compensators for multivariable linear systems. The main contribution is that the formulation treats a long-standing problem in optimal output feedback, that of providing a rationale for properly selecting the plant and compensator-state weightings, and the initial condition distribution matrix, to achieve both performance and robustness. The examples illustrate the effectiveness of the formulation and overall design methodology.

## Acknowledgments

This research was partially supported by Grant NAG-1-243 from NASA Langley Research Center and by Contract E16-A02 from the Army Research Office.

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